Notes 3. UNIFORM CONVERGENCE

Uniform convergence is the main theme of this chapter. In Section 1 pointwise and uniform convergence of sequences of functions are discussed and examples are given. In Section 2 the three theorems on exchange of pointwise limits, integration and differentiation which are corner stones for all later development are proven. They are reformulated in the context of infinite series of functions in Section 3. The last two important sections demonstrate the power of uniform convergence. In Sections 4 and 5 we introduce the exponential function, sine and cosine functions based on differential equations. Although various definitions of these elementary functions were given in more elementary courses, here the definitions are the most rigorous one and all old ones should be abandoned. Once these functions are defined, other elementary functions such as the logarithmic function, power functions, and other trigonometric functions can be defined accordingly. A notable point is at the end of the section, a rigorous definition of the number π is given and showed to be consistent with its geometric meaning.

3.1 Uniform Convergence of Functions

Let E be a (non-empty) subset of $\mathbb{R}$ and consider a sequence of real-valued functions $\{f_n\}, n \geq 1$ and f defined on E. We call $\{f_n\}$ **pointwisely converges** to f on E if for every $x \in E$, the sequence $\{f_n(x)\}$ of real numbers converges to the number $f(x)$. The function f is called the **pointwise limit** of the sequence. According to the limit of sequence, pointwise convergence means, for each $x \in E$, given $\varepsilon > 0$, there is some $n_0(x)$ such that

$$|f_n(x) - f(x)| < \varepsilon , \quad \forall n \geq n_0(x) .$$

We use the notation $n_0(x)$ to emphasis the dependence of $n_0(x)$ on ε and x. In contrast, $\{f_n\}$ is called **uniformly converges** to f if $n_0(x)$ can be chosen to be independent of x, that is, uniform in x. In other words, it means, given $\varepsilon > 0$, there is some n_0 such that

$$|f_n(x) - f(x)| < \varepsilon , \quad \forall n \geq n_0, \ x \in E .$$

We shall use the notation $f_n \rightrightarrows f$ to denote $\{f_n\}$ uniformly converges to f.

Example 3.1 Consider the sequence of functions $\{x^n\}$ defined on $[0, 1]$. Its pointwise limit is easily found. Indeed, when $x \in (0, 1), x^n \to 0$ as $n \to \infty$ and, when $x = 1, x^n \to 1$ as $n \to \infty$. We see that the pointwise limit of this sequence is the function $\psi(x) = 0, x \in [0, 1)$ and $\psi(1) = 1$. Next, we claim that this sequence is not uniform convergent. Indeed, for $x \in [0, 1), x^n = |x^n - 0| < \varepsilon$

iff $n > \log \varepsilon / \log x$. It follows that $n_0(x) > \log \varepsilon / \log x$. As x comes close to 1, $n_0(x)$ becomes unbounded. Therefore, there is no way to find an n_0 to make $|x^n - 0| < \varepsilon$, $n \geq n_0$, for all $x \in [0, 1)$.

Example 3.2 Let φ be a function which is positive and continuous on $[1/2, 3/4]$ and 0 elsewhere. Define $f_n(x) = \varphi(x - n)$. It is clear that it converges point-wisely to the zero function. In fact, given $x > 0$ we can find some N such that $x \in [N, N + 1]$. Taking $\varepsilon < 1$, From $|f_n(x) - 0| = |\varphi(x - n)| < \varepsilon$ if and only if $n \geq N + 1$. That is, $n_0(x) \geq N + 1 \to \infty$ as $x \to \infty$. We conclude that the convergence is not uniform.

The following fact is immediate from the definition, but is worthwhile to single out.

Proposition 3.1. *Suppose that $\{f_n\}$ converges uniformly to f on E. Then $\{f_n\}$ converges pointwisely to f on E.*

Example 3.3 Consider the functions $k_n(x) = \cos[n\pi/(1+x^2)]$ on $[-1, 1]$. At each x, $k_n(x)$ keeps jumping up and down and becomes more rapidly as n increases. We do not see any possible limit. This suggests that $\{k_n\}$ is not convergent. In fact, we focus at the point $x = 0$ where $k_n(0) = \cos n\pi = (-1)^n$ does not have a limit. So this sequence is not even pointwise convergent, let alone uniform convergent.

Observe that to establish uniform convergence it suffices to restrict ε to some interval $(0, \varepsilon_0]$. In the following examples we implicitly assume $\varepsilon \in (0, 1)$.

Example 3.4. Let $f_n(x) = 1/(n^2 + x^2)$, $x \in \mathbb{R}$, $n \in \mathbb{N}$. By plotting graphs it is easily seen that f_n's tend to zero nicely. We guess that the zero function is their uniform limit. To prove this, we note the following simple estimate

$$|f_n(x)| \leq \frac{1}{n^2}, \quad \forall x \in \mathbb{R}.$$

Therefore, for $\varepsilon > 0$, by taking $n_0 > 1/\sqrt{\varepsilon}$,, we have $|f_n(x) - 0| < \varepsilon$ for all $n \geq n_0$ and $x \in \mathbb{R}$. So $f_n \rightrightarrows 0$.

Example 3.5. Let $g_n(x) = (x^2 + 1)e^{x/n}$, $x \in [0, 1]$. As $n \to \infty$, $x/n \to 0$ for all $x > 0$. It suggests that $\{g_n\}$ tends to $g(x) \equiv (x^2 + 1)$ as $n \to \infty$. To prove it we observe that $|g_n(x) - g(x)| = |(x^2 + 1)(e^{x/n} - 1)| \leq 2|e^{x/n} - 1|$ on $[0, 1]$. For $x \in [0, 1]$, we know that $0 \leq e^{x/n} - 1 \leq e^{1/n} - 1$. As $\lim_{n \to \infty} e^{1/n} = 1$, for $\varepsilon > 0$, there exists some n_0 such that $e^{1/n} - 1 < \varepsilon$ for all $n \geq n_0$. It follows that

$|g_n(x) - g(x)| < \varepsilon, \forall x \in [0,1], n \geq n_0$, that is, $g_n \rightrightarrows g$ on $[0,1]$.

The collective convergence behavior of a sequence of functions can be described in terms of a single numerical sequence. Introduce the **supnorm** (or **uniform norm**) of a function g by letting

$$\|g\| = \sup\{|g(x)| : x \in E\} .$$

It is clear that $\|g\|$ is a finite number if and only if g is a bounded function on E. The following properties of the sup-norm are evident and will be used from time to time.

Proposition 3.2. *Let f, g be bounded functions on E. Then*

(a)
$$\|f\| \geq 0 \text{ and } \|f\| = 0 \text{ iff } f(x) = 0 , \forall x \in E .$$

(b)
$$\|\alpha f\| = |\alpha| \|f\| , \alpha \in \mathbb{R} .$$

(c)
$$\|f + g\| \leq \|f\| + \|g\| .$$

We observe that uniform convergence of $\{f_n\}$ is equivalent to the convergence of the sequence $\{\|f_n\|\}$.

Proposition 3.3. *Let $\{f_n\}$ be defined on E with pointwise limit f. Then $\{f_n\}$ converges uniformly to f if and only if $\lim_{n\to\infty} \|f_n - f\| = 0$.*

Proof. Let $\{f_n\}$ converge uniformly to f. For $\varepsilon > 0$, there is some n_0 such that $|f_n(x) - f(x)| < \varepsilon/2$ for all $n \geq n_0$ and $x \in E$. Taking supremum over all $x \in E$, we get
$$\|f_n - f\| = \sup_x |f_n(x) - f(x)| \leq \frac{\varepsilon}{2} < \varepsilon , \quad n \geq n_0 .$$
The converse is evident. $\qquad\qquad\square$

For $\varepsilon > 0$, the ε-**tube** of f is the set in the plane given by

$$\{(x,y) : f(x) - \varepsilon < y < f(x) + \varepsilon, \ x \in E\} .$$

Geometrically, that $\{f_n\}$ converges uniformly to f means for each $\varepsilon > 0$, there is some n_0 such that all graphs of $f_n, n \geq n_0$ lie inside the ε-tube of f.

Example 3.6. Consider $h_n(x) = x/(n^2 + x^2)$ on $\mathbb{R}$. By comparing the order of growth of the numerator and denominator at infinity, one is convinced that h_n tends to 0 in certain sense. Instead of working out an estimate (which is now

not so straightforward) as above, we argue by evaluating the supnorm directly. Observing that the supremum of the absolute value of a function is equal to either the maximum or the negative of the minimum of the function (depending on which one has larger corresponding value), we differentiate each h_n to find its maximum/minimum. By setting

$$h'_n(x) = \frac{(n^2 + x^2) - 2x^2}{(n^2 + x^2)^2} = 0,$$

we find that there are two critical points $x = n, -n$. It is not hard to see that the former is the maximum and the latter minimum, and $\|h_n\| = |h(\pm n)| = 1/2n \to 0$ as $n \to \infty$.

From these examples you can see that the study of uniform convergence of sequences of functions $\{f_n\}$ requires certain effort. Summarizing what have been done, we have

- Determine the pointwise limit of the sequence of functions. It is not pointwise convergent (hence not uniformly convergent) when the pointwise limit does not exist somewhere, that is, the sequence diverges at some point, see Example 3.3.

- Use various methods to estimate $|f_n(x) - f(x)|$ independent of x, see Example 3.4 and Example 3.5.

- Finally, if possible, evaluate the supnorm $\|f_n - f\|$ directly by the method of differentiation, see Example 3.6.

A basic property of $\mathbb{R}^n$ is that all Cauchy sequences converge in $\mathbb{R}^n$. It is useful for the establishment of the convergence of a sequence when its limit is not known. The concept of a Cauchy sequence makes perfect sense here. We call a sequence of functions $\{f_n\}$ on E a **Cauchy sequence (in supnorm)** if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\|f_n - f_m\| < \varepsilon, \qquad \forall n, m \geq n_0.$$

Just the same as in the Euclidean space, we have

Theorem 3.4. *Let $\{f_n\}$ be a sequence of functions on E. It is uniformly convergent if and only if it is a Cauchy sequence in supnorm.*

Proof. " $\Rightarrow$ ". Let $\{f_n\}$ converge to f in the supnorm. For $\varepsilon > 0$, there is some n_0 such that $\|f_n - f\| < \varepsilon/2$ for all $n \geq n_0$. By the triangle inequality (see

Proposition 3.2)

$$\|f_n - f_m\| \le \|f_n - f\| + \|f - f_m\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \forall n \ge n_0,$$

so $\{f_n\}$ is a Cauchy sequence in supnorm.

" $\Leftarrow$ ". Let $\{f_n\}$ be a Cauchy sequence in supnorm. For $\varepsilon > 0, \exists n_0 \in \mathbb{N}$ such that

$$\|f_n - f_m\| < \frac{\varepsilon}{2}, \quad \forall n, m \ge n_0.$$

In other words, for all $x \in E$,

$$|f_n(x) - f_m(x)| < \frac{\varepsilon}{2}, \quad \forall n, m \ge n_0.$$

It shows that $\{f_n(x)\}$ is a numerical Cauchy sequence, so it converges to a real number y_x. We define a function f by setting $f(x) = y_x$. Then for each x, $\{f_n(x)\}$ converges to $f(x)$ as $n \to \infty$. Now, by passing $m \to \infty$ above,

$$\|f_n - f\| = \sup_{x \in E} |f_n(x) - f(x)| \le \frac{\varepsilon}{2} < \varepsilon, \quad \forall n \ge n_0,$$

whence $f_n \rightrightarrows f$. $\qquad\square$

The Cauchy sequence test may be regarded as the ultimate test for uniform convergence. It works especially when the limit function is no way to find. Let us examine the following example.

Example 3.7 Consider the sequence of the functions given by

$$\varphi_n(x) = \sum_{j=1}^{n} \frac{\sin jx}{j^2}, \ n \ge 1.$$

Apparently there is no simple way to find the pointwise limit of this sequence. However, using the boundedness of the sine function, we have

$$|\varphi_n(x) - \varphi_m(x)| < \sum_{m+1}^{n} \frac{1}{j^2}.$$

As $\sum_{1}^{\infty} 1/j^2 < \infty$, for each ε we can find some n_0 such that $\sum_{m+1}^{n} j^{-2} < \varepsilon/2$. It follows that
$$\|\varphi_n - \varphi_m\| = \sup_{x} |\varphi_n(x) - \varphi_m(x)| \le \frac{\varepsilon}{2} < \varepsilon.$$

we conclude that for $n, m \ge n_0$, $\{\varphi_n\}$ forms a Cauchy sequence in supnorm in $\mathbb{R}$. By Theorem 3.4 it converges uniformly to some function.

The advantage of the Cauchy criterion for uniform convergence is that we do not need any information on the limit function before hand. Many applications will make this point crystal clear in our subsequent development. This criterion is very general. In fact, we do not have to specify whether the functions are bounded or continuous on E. It makes sense for *any sequence* of functions on E and the limit is again a function defined on E.

3.2   Interchange of Limits

Uniform convergence turns out to be an indispensable notion in analysis. Many properties which are lost under the process of taking pointwise limit are preserved under uniform limit. In this section we study how continuity, differentiability and integrability are preserved under various uniform convergence assumptions. They will have greater applications in our subsequent development.

First of all, we have

Theorem 3.5. *Let $\{f_n\}$ be a sequence of functions which converges uniformly to the function f on E.*

(a) If $\{f_n\} \subset B(E)$, then $f \in B(E)$.

(b) If $\{f_n\} \subset C(E)$, then $f \in C(E)$.

Here we have used $B(E)$ to denote all bounded functions and $C(E)$ all continuous functions on E.

Proof.   (a) taking $\varepsilon = 1$, there is some n_0 such that $|f_n(x) - f(x)| < 1$ for all $n \geq n_0$ and $x \in E$. Therefore,

$$|f(x)| \leq |f(x) - f_{n_0}(x)| + |f_{n_0}(x)| \leq 1 + \sup_x |f_{n_0}(x)| \ ,$$

so $\|f\| = \sup_x |f(x)| \leq 1 + \|f_{n_0}\|$ is finite.

(b) When every f_n is continuous, we claim that f is also continuous. For, we take $\varepsilon > 0$ to be arbitrary and fix n_1 such that $|f_n(x) - f(x)| < \varepsilon/3$ for $n \geq n_1$. Let $x_0 \in E$. As f_{n_1} is continuous, there exists $\delta > 0$ such that

$$|f_{n_1}(x) - f_{n_1}(x_0)| < \frac{\varepsilon}{3}, \quad \text{if } |x - x_0| < \delta, \ x \in E.$$

Therefore, for $x \in E$, $|x - x_0| < \delta$,

$$|f(x) - f(x_0)| \leq |f(x) - f_{n_1}(x)| + |f_{n_1}(x) - f_{n_1}(x_0)| + |f_{n_1}(x_0) - f(x_0)|$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$
$$= \varepsilon,$$

which shows that f is continuous at x_0. Since $x_0 \in E$ is arbitrary, it follows that $f \in C(E)$.

$\square$

Example 3.1 revisited. We considered the functions $\{x^n\}$ on $[0, 1)$ and showed that it is not uniformly convergent to 0. Now, let us consider it on $[0, 1]$. The sequence converges pointwisely to the function $\psi(x)$ which is given by $\psi(x) = 0$, $x \in [0, 1)$ and $\psi(1) = 1$, but the convergence is not uniform (it is not on $[0, 1)$, therefore not on any larger domain). Each x^n is continuous on $[0, 1]$ but ψ has a discontinuity at $x = 0$. This example shows that the pointwise limit of a sequence of continuous functions may not be continuous.

In MATH2050/2060 we have learnt three types of limit processes:

- First, limits of functions $\lim_{x \to x_0} f(x)$.

- Second, differentiation

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \ .$$

- Third, integration

$$\int_a^b f = \sum_{||P|| \to 0} f(z_j) \Delta x_j \ .$$

Our first result is concerned with the interchange of limits which, in symbolic form, can be expressed as

$$\lim_{x \to x_0} \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \lim_{x \to x_0} f_n(x). \tag{3.1}$$

Theorem 3.6. *Let $\{f_n\} \subseteq C(E)$ converge uniformly to f on E. Then for every $x_0 \in E$, (3.1) holds.*

Proof. This is essentially Theorem 3.5(b). The left hand side of (3.1) is $\lim_{x \to x_0} f(x)$ which, by the continuity of f is equal to $f(x_0)$. On the other hand, the right hand side of (3.1) is equal to $\lim_{n \to \infty} f_n(x_0)$, which by the pointwise convergence of f_n, is also equal to $f(x_0)$, done. $\square$

Next, we consider the situation

$$\int_a^x \lim_{n\to\infty} f_n(t)dt = \lim_{n\to\infty} \int_a^x f_n(t)dt \tag{3.2}$$

Theorem 3.7. *Let $\{f_n\} \subseteq R[a,b]$ converge uniformly to f. Then $f \in R[a,b]$. Moreover, the indefinite integrals of f_n's converge uniformly to the indefinite integral of f. In particular, (3.2) holds for all $x \in [a,b]$.*

Proof. From the definition of integrability, all f_n's are bounded, so f is also bounded as their uniform limit by Proposition 3.5(a). By uniform convergence, given $\varepsilon > 0$, there exists n_0 such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{4(b-a)}, \quad \forall n \geq n_0, \text{ and } \forall x \in [a,b].$$

It follows that

$$
\begin{aligned}
\operatorname{osc}_I(f_n - f) &\leq \sup_{x,y\in[a,b]} |f_n(x) - f(x) - (f_n(y) - f(y))| \\
&\leq \sup_x |f_n(x) - f(x)| + \sup_y |f_n(y) - f(y)| \\
&\leq \frac{\varepsilon}{2(b-a)},
\end{aligned}
$$

on every subinterval I of $[a,b]$. Now, as f_{n_0} is integrable, we can find a partition P such that $\sum_P \operatorname{osc}_{I_j} f_{n_0} \Delta x_j < \varepsilon/2$. Therefore, we have

$$
\begin{aligned}
\sum_P \operatorname{osc}_{I_j} f \Delta x_j &\leq \sum_P \operatorname{osc}_{I_j} f_{n_0} \Delta x_j + \sum_P \operatorname{osc}_{I_j}(f - f_{n_0})\Delta x_j \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2(b-a)}(b-a) \\
&= \varepsilon.
\end{aligned}
$$

So $f \in R[a,b]$ by the Second Integrability Criterion.

Next, denote the indefinite integrals of f_n and f by F_n and F respectively.

For all $x \in [a, b]$, we have

$$
\begin{aligned}
|F_n(x) - F(x)| &= \left| \int_a^x (f_n(t) - f(t)) dt \right| \\
&\leq \int_a^x |f_n(t) - f(t)| dt \\
&\leq \int_a^b |f_n(t) - f(t)| dt \\
&\leq (b-a) \|f_n - f\| \\
&< \frac{\varepsilon}{4},
\end{aligned}
$$

for $n \geq n_0$. Hence F_n converges uniformly to F. Now (3.2), which asserts that F_n converges pointwisely to F, follows from Proposition 3.1.

$\square$

Example 3.8. Here we show that the uniform convergence of $\{f_n\}$ cannot be replaced by pointwise convergence in Theorem 3.7. Let $\varphi_n(x)$ be the function on $[0, 1]$ which is equal to $n^2 x$, $x \in [0, 1/n]$, equal to $-n^2(x - 2/n)$, $x \in [1/n, 2/n]$ and becomes zero elsewhere. It is clear this sequence converges pointwisely but not uniformly to the zero function on $[0, 1]$. We have

$$
\int_0^1 \varphi_n(x) dx = 1 \neq \int_0^1 0 \, dx = 0 \ .
$$

Our last result is concerned with

$$
\frac{d}{dx} \lim_{n \to \infty} f_n = \lim_{n \to \infty} \frac{d}{dx} f_n. \tag{3.3}
$$

Theorem 3.8. *Let $\{f_n\}$ be a sequence of differentiable functions on some interval I. Assume that*

(a) it converges to a function f pointwisely on I; and

(b) $\{f_n'\}$ converges uniformly to a function g on I.

Then f is differentiable and $f' = g$ on I.

Proof. Fix $x_0 \in I$. For each $x \in I \setminus \{x_0\}$, by applying Mean-Value Theorem to the function $f_n - f_m$ we find z between x and x_0 such that

$$
(f_n - f_m)(x) - (f_n - f_m)(x_0) = (x - x_0)(f_n'(z) - f_m'(z)).
$$

Let $\varepsilon > 0$. As $f'_n \rightrightarrows g$ on $[a, b]$, $\{f'_n\}$ is a Cauchy sequence in supnorm. There exists $n_0 \in \mathbb{N}$ such that $||f'_n - f'_m|| < \varepsilon/3$ for all $n, m \geq n_0$. Therefore, we have

$$\left| \frac{f_n(x) - f_n(x_0)}{x - x_0} - \frac{f_m(x) - f_m(x_0)}{x - x_0} \right| \leq ||f'_n - f'_m|| < \frac{\varepsilon}{3}.$$

Letting $m \to \infty$ in this estimate we obtain

$$\left| \frac{f_n(x) - f_n(x_0)}{x - x_0} - \frac{f(x) - f(x_0)}{x - x_0} \right| \leq \frac{\varepsilon}{3}, \quad \forall n \geq n_0. \tag{3.4}$$

As $\{f'_n\} \rightrightarrows g$, we can fix a large $N \geq n_0$ so that

$$|f'_N(y) - g(y)| < \frac{\varepsilon}{3}, \quad \forall y \in I. \tag{3.5}$$

As f_N is differentiable at x_0, there exists some $\delta > 0$ such that

$$\left| \frac{f_N(x) - f_N(x_0)}{x - x_0} - f'_N(x_0) \right| < \frac{\varepsilon}{3}, \quad \text{whenever } 0 < |x - x_0| < \delta. \tag{3.6}$$

By combining (3.4), (3.5) and (3.6), we have, for any $x \in I$ with $0 < |x - x_0| < \delta$,

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - g(x_0) \right| \leq \left| \frac{f(x) - f(x_0)}{x - x_0} - \frac{f_N(x) - f_N(x_0)}{x - x_0} \right|$$
$$+ \left| \frac{f_N(x) - f_N(x_0)}{x - x_0} - f'_N(x_0) \right| + \left| f'_N(x_0) - g(x_0) \right|$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon.$$

So $f'(x_0)$ exists and is equal to $g(x_0)$. Since $x_0 \in I$ could be any point in I, f' exists and is equal to g on I. $\qquad \square$

Remark. In this theorem we can replace (a) with a weaker assumption, namely,

(a)' $\{f_n(x_0)\}$ is convergent at some $x_0 \in I$.

To see that (a)' and (b) imply (a), we let $y_0 = \lim_{n \to \infty} f_n(x_0)$ and define

$$f(x) = y_0 + \int_{x_0}^{x} g(t) dt, \quad x \in I.$$

As $f'_n \rightrightarrows g$, it is easy to see or simply by Theorem 3.87owever, (a) and (b) are easier to memorize.

Example 3.9. Consider $\varphi_n(x) = x^{n+1}/(n+1), n \geq 1$. It is clear that $\{\varphi_n\}$ con-

verges pointwisely (in fact, uniformly) to $\varphi(x) \equiv 0$. On the other hand, $\varphi'_n = x^n$ given in Example 3.4 revisited. We knew that $\{x^n\}$ converges to the discontinuous function ψ pointwisely but not uniformly on $[0, 1]$. The limit function φ is differentiable on $[0, 1]$ and it is not equal to ψ. This shows that the assumption "$f'_n \rightrightarrows g$" cannot be replaced by pointwise convergence in Theorem 3.8.

The next result may be regarded as a special one. It shows pointwise convergence together with some monotonicity implies uniform convergence.

Theorem 3.9 (**Dini's Theorem**). *Suppose that $\{f_n\}$ is a monotone sequence of continuous functions on $[a, b]$. Suppose that it converges pointwisely to the continuous function f on $[a, b]$. Then $\{f_n\}$ converges uniformly to f.*

A sequence $\{f_n\}$ is monotone means either $\{f_n(x)\}$ is increasing or decreasing at every x.

Proof. Let us take f_n to be decreasing and so $f_n - f$ decreases to 0 pointwisely. If $f_n - f$ does not converge to 0 uniformly, by definition there exists some ε_0 such that $\|f_n - f\| \geq 2\varepsilon_0 > 0$ for infinitely many n's. For simplicity we may assume that it is so for all n's. Then we can find, for each n, a point x_n in $[a, b]$ such that

$$f_n(x_n) - f(x_n) = |f_n(x_n) - f(x_n)| \geq \varepsilon_0.$$

By Bolzano-Weierstrass theorem, $\{x_n\}$ contains a subsequence $\{x_{n_j}\}$ convergent to some x^*. As each f_n is decreasing,

$$f_m(x) - f(x) \geq f_n(x) - f(x), \ \forall m, n, \ m \leq n.$$

Taking $m = n_k$ and $n = n_j, j \geq k$, we obtain

$$f_{n_k}(x_{n_j}) - f(x_{n_j}) \geq f_{n_j}(x_{n_j}) - f(x_{n_j}) \geq \varepsilon_0.$$

Now fix n_k and let n_j go to infinity, we get

$$f_{n_k}(x^*) - f(x^*) \geq \varepsilon_0,$$

for all n_k, but this impossible because $f_n - f$ tends to 0 pointwisely. This contradiction shows that the convergence must be uniform. $\qquad\Box$

3.3   Series of Functions

As we learnt before, convergence of an infinite series of numbers means the convergence of the sequence of numbers formed by the partial sums of the series. From the relation between a series and its sequence of partial sums, many results on sequences can be reformulated as results for series. For the case of sequences

of functions there is no exception.

Given a series of functions on some $E \subseteq \mathbb{R}$, we use the notation $\sum_{j=1}^{\infty} f_j$ to denote the series. The n-th partial sum of this series is given by the function $s_n(x) \equiv \sum_{j=1}^{n} f_j(x)$. We call the infinite series $\sum_{j=1}^{\infty} f_j$ **converges uniformly** (resp. **converges pointwisely**) if the sequence of partial sums, $\{s_n\}$, converges uniformly (resp. pointwisely) on E. The limit function is usually denoted also by $\sum_{j=1}^{\infty} f_j$.

Thus there are two meanings for $\sum_{j=1}^{\infty} f_j(x)$ or $\sum_j f_j$: First, it is simply a notation standing for the infinite series formed by f_j's. Second it means the limit $\lim_{n \to \infty} s_n(x)$ provided $\{s_n(x)\}$ is a convergent sequence.

Without much effort, we transplant Theorems 3.6, 3.7 and 3.8 to series of functions.

Theorem 3.6' *Let $\sum_{j=1}^{\infty} f_j$ be a series of continuous functions on E which converges uniformly. Then its limit $\sum_{j=1}^{\infty} f_j$ is also continuous on E.*

Proof. Let $\{s_n(x)\}$ be the sequence of partial sums of the series $\sum_{j=1}^{\infty} f_j(x)$. By assumption each f_j is continuous, so is each s_n. Since $s_n \rightrightarrows \sum_{j=1}^{\infty} f_j$, by Theorem 3.6, $\sum_{j=1}^{\infty} f_j$ is continuous. $\square$

Theorem 3.7' *Let $\sum_{j=1}^{\infty} f_j$ be a series of Riemann integrable functions which converges uniformly on $[a, b]$. Then its limit $\sum_{j=1}^{\infty} f_j$ is also integrable on $[a, b]$, and*

$$\sum_{j=1}^{\infty} \int_a^x f_j = \int_a^x \sum_{j=1}^{\infty} f_j, \quad \forall x \in [a, b],$$

holds.

Proof. We have the partial sums $\{s_n\}$ of f_n's converges uniformly to the function $\sum_{j=1}^{\infty} f_j$. From Theorem 3.7 we have $\int_a^x s_n$ converges uniformly to $\int_a^x \sum_{j=1}^{\infty} f_j$. Thus,

$$\sum_{j=1}^{\infty} \int_a^x f_j = \lim_{n \to \infty} \sum_{j=1}^{n} \int_a^x f_j = \lim_{n \to \infty} \int_a^x s_n = \int_a^x \sum_{j=1}^{\infty} f_j.$$

$\square$

Theorem 3.8' *Let $\sum_{j=1}^{\infty} f_j$ be a series of differentiable functions on an interval I which converges pointwisely on I. Suppose that $\sum_{j=1}^{\infty} f_j'$ converges uniformly*

on I. Then $\sum_{j=1}^{\infty} f_j$ converges uniformly to a differentiable function on I, and

$$\frac{d}{dx} \sum_{j=1}^{\infty} f_j = \sum_{j=1}^{\infty} \frac{df_j}{dx}.$$

I leave the proof of this theorem as an exercise.

In contrast to sequences of functions, we have the following criterion of uniform convergence tailored for series of functions. It is perhaps the most useful one.

Theorem 3.10 (**Weierstrass M-test**). *Let* $\{f_n\}, n \in \mathbb{N}$*, be functions defined on* E*. Suppose there exist non-negative numbers* a_n*'s satisfying*

$$|f_n(x)| \leq a_n, \quad \forall x \in E, \quad \forall n \geq N \text{ for some } N.$$

Then the series $\sum_{n=1}^{\infty} f_n$ *is uniformly convergent on* E *provided* $\sum_{n=1}^{\infty} a_n$ *is convergent.*

Proof. For simplicity we take $N = 1$. Denote by s_n the n-th partial sum of $\sum_{j=1}^{\infty} f_j$. Let $\varepsilon > 0$. As $\sum a_n < \infty$, there exists an $n_0 \in \mathbb{N}$ such that

$$\left| \sum_{j=1}^{n} a_j - \sum_{j=1}^{m} a_j \right| = \sum_{j=m+1}^{n} a_j < \varepsilon, \quad \forall n, m \geq n_0 \text{ with } n > m.$$

As each $f_n(x)$ is dominated by a_n, we have

$$
\begin{aligned}
|s_n(x) - s_m(x)| &= \left| \sum_{j=m+1}^{n} f_j(x) \right| \\
&\leq \sum_{j=m+1}^{n} |f_j(x)| \\
&\leq \sum_{j=m+1}^{n} a_j \\
&< \varepsilon, \quad \forall m, n \geq n_0 ,
\end{aligned}
$$

on E. In other words, $\{s_n\}$ is a Cauchy sequence in supnorm, hence it converges uniformly according to Theorem 3.4. $\square$

Example 3.11. (a) Consider $\{f_n\}$ where $f_n(x) = \cos nx / n^2$, for $x \in \mathbb{R}, n \in \mathbb{N}$. It is clear $|f_n(x)| \leq 1/n^2$ for all $x \in \mathbb{R}$. By the Weierstrass M-test we conclude that the series $\sum_{n=1}^{\infty} \cos nx / n^2$ converges uniformly to a continuous function ϕ

on the real line. Usually we simply let

$$\sum_{j=1}^{\infty} \frac{\cos nx}{n^2}$$

to denote the limit function ϕ.

(b) Consider the series $\sum_{n=1}^{\infty} e^{-nx}$, $x \in [1, \infty)$. We have $|e^{-nx}| \leq e^{-n}$ for $x \geq 1$. Since $e^n \geq n^2/2$, $e^{-n} \leq 2n^{-2}$. By $\sum_{n=1}^{\infty} n^{-2} < \infty$, M-Test and Theorem 3.6', this series converges uniformly to a continuous function. In fact, we claim that this function is differentiable. To see this it suffices to examine the series obtained by differentiating the original series, which is given by $\sum_{n=1}^{\infty} (-n)e^{-nx}$. Using $e^n \geq n^3/3!$, we have $ne^{-n} \leq 6n^{-2}$. Therefore by M-Test and Theorem 3.7', this derived series converges uniformly to a continuous function on $[1, \infty)$. Then an application of Theorem 3.8' shows that the original limit function $\sum_{n=1}^{\infty} e^{-nx}$ is differentiable on $[1, \infty)$ with derivative given by $-\sum_{n=1}^{\infty} ne^{-nx}$. Actually, by repeating this argument one can show that $\sum_{n=1}^{\infty} e^{-nx}$ is a smooth function.

3.4   The Exponential and Logarithmic Functions

The most familiar functions are the constants and linear functions. By multiplying and adding them up we get polynomials. Next by taking quotients rational functions come into play. Using the existence of inverse to the function $x \mapsto x^m$ one can form the m-th root of a rational function. We may also form new functions by taking composition of two functions. By repeating these operations we obtain many many functions. We know how to differentiate these functions using those familiar rules of differentiation. However, there are functions which cannot be obtained by finitely many steps of these operations. So they are called transcendental. Among many, the exponential function and the cosine/sine functions are the most important ones. The former describes natural growth and the latter depicts periodic motions. Closely associated are the logarithmic function and other trigonometric functions. In these notes we define these transcendental elementary functions rigorously and derive their basic properties. As an application we use the exponential and logarithmic functions to define any real power of a positive number; so far we have only defined it for rational powers.

We emphasis that the trigonometric functions we learnt in high school were defined by geometric method. We do not take them as rigor ones. Here we shall define the cosine and sine functions to be certain solutions to some differential equations. In order to be systematic, the same approach is also adopted for the exponential function. The advantage of this approach will soon become evident.

Our first elementary transcendental function is the exponential function. Consider the differential equation

$$\frac{df}{dx} = f, \quad f(0) = 1. \tag{3.7}$$

We would like to find a function f solving this problem. It is not hard to come up with a formal one. Letting

$$f(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

then $f(0) = 1$ and

$$f'(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

so (3.7) is satisfied.

Theorem 3.11.

(a) *There exists a unique solution E to (3.7), In fact, for each $x \in \mathbb{R}$, $E(x)$ is given by the series $\sum_{j=0}^{\infty} x^j/j!$ where the convergence is uniform on every bounded interval.*

(b) *E is smooth on $\mathbb{R}$.*

(c) *E is positive on $\mathbb{R}$, $\lim_{x \to \infty} E(x) = \infty$ and $\lim_{x \to -\infty} E(x) = 0$. In particular, its range is $(0, \infty)$.*

(d) *E is strictly increasing and strictly convex.*

This solution E is called the **exponential function**.

Proof. (a) Let $E(x)$ be the infinite series $\sum_{j=0}^{\infty} \frac{x^j}{j!}$. We first show that it converges uniformly on each interval $[-M, M]$ for any $M > 0$. To this end, we use M-test. Let

$$a_j = \frac{M^j}{j!}.$$

It is an old exercise that $\sum_{j=0}^{\infty} a_j < \infty$ for any fixed M. As

$$\left| \frac{x^j}{j!} \right| \leq a_j, \quad \forall\, x \in [-M, M],$$

by Weierstrass M-test, $\sum_{j=0}^{\infty} x^j/j!$ converges uniformly on $[-M, M]$ to $E(x)$ and by Theorem 3.6', E is a continuous function on $[-M, M]$. Next, we observe that

the series obtained by termwise differentiating $E(x)$ is given by $\sum_{j=1}^{\infty} jx^{j-1}/j!$ which is the same as $E(x) = \sum_{j=0}^{\infty} x^j/j!$. It converges uniformly on $[-M, M]$. By Theorem 3.8' that E is differentiable and satisfies (3.7) on $[-M, M]$. Since $M > 0$ is arbitrary, it follows that (3.7) holds on $\mathbb{R}$.

To prove uniqueness let f_1 and f_2 be two solutions to (3.7). Then their difference $f = f_2 - f_1$ satisfies the equation in (3.7) and $f(0) = 0$, and $f_1 \equiv f_2$ iff $f \equiv 0$. In the following we will establish a slightly general result; we replace 0 be an arbitrary point x_0.

Let f satisfy the equation in (3.7) and $f(x_0) = 0$ at some x_0. Then $f \equiv 0$.

Indeed, it is sufficient to show it vanishes for all $x \in [x_0 - 1/2, x_0 + 1/2]$, for we may apply the assertion replacing x_0 by $x_0 \pm 1/2$ and then spread out. To achieve that, we integrate the equation to get

$$
\begin{aligned}
f(x) &= f(x_0) + \int_{x_0}^{x} f'(s)ds \\
&= f(x_0) + \int_{x_0}^{x} f(s)ds \ .
\end{aligned}
$$

For $z \in (0, 1/2)$, let $|f(x_1)| = \max\{|f(x)| : x \in [x_0 - 1/2, x_0 + 1/2]\}$. We have

$$
\begin{aligned}
|f(x_1)| &\leq \left| \int_{x_0}^{x_1} |f(s)|ds \right| \\
&\leq |f(x_1)| \left| \int_{x_0}^{x_1} ds \right| \\
&= \frac{|f(x_1)|}{2},
\end{aligned}
$$

which forces $|f(x_1)| = 0$, in other words, f vanishes identically on $[x_0 - 1/2, x_0 + 1/2]$.

(b) This follows from repeatedly using the equation in (3.7), which asserts that whenever the right side f is k-times differentiable implies the left hand side f' is k-times differentiable, so f is $k+1$-times differentiable. Repeating this reasoning we see that E is infinitely many times differentiable, that is, it is smooth.

(c) From $E(x) = \sum_{j=0}^{\infty} x^j/j! \geq 1 + x$, $\forall x > 0$, it shows that $E(x) \to \infty$ as $x \to \infty$. Since $E > 0$ on $[0, \infty)$, if it is not positive on $\mathbb{R}$, there exists $x_0 < 0$ so that $E(x_0) = 0$. However, this is impossible in view of the proof of uniqueness above. We shall prove $E(x) \to 0$ as $x \to -\infty$ later. Assuming this fact, it follows from the intermediate value theorem that $E(\mathbb{R}) = (0, \infty)$.

(d) From $E'' = E' = E > 0$ we get (d).

Now, we show that $E(x) \to 0$ as $x \to -\infty$. As E is strictly increasing and always positive, $\lim\limits_{x \to -\infty} E(x) = \alpha \geq 0$ exists. For $x < 0$,

$$1 = E(0) = E(x) + \int_x^0 E'(t)\, dt$$

$$\geq 0 + \int_x^0 E(t)\, dt > \alpha(-x) \to \infty, \text{ as } x \to -\infty,$$

which forces $\alpha = 0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We point out that in fact Theorem 3.11(c) concerning $E(x)$ as $x \to \infty$ can be sharpened to $\lim_{x \to \infty} E(x)/x^n = \infty$ for every $n \geq 1$. In particular, it implies the basic fact that the exponential function grows faster than any polynomial at ∞. Indeed, it suffices to observe the inequality

$$E(x) \geq \frac{x^n}{n!}$$

for every n and $x \geq 0$.

Now we establish the most important properties of the exponential function. It demonstrates the power of the approach by differential equations.

Theorem 3.12.
  (a)
$$E(x + y) = E(x)E(y), \quad \forall x, y \in \mathbb{R},$$

  (b)
$$E(\alpha x) = E(x)^\alpha, \quad \forall \alpha \in \mathbb{Q}.$$

Proof. (a) Let $f(x) = E(x + y)/E(y)$ when y is fixed. One readily verifies that f satisfies (3.7), so by uniqueness, $f(x) = E(x)$.
(b) It follows from (a) and induction that for any $x \in \mathbb{R}$ and $n \in \mathbb{N}$, $E(nx) = E(x)^n$. Thus,

$$E(x) = E \bigg(\overbrace{\frac{x}{n} + \cdots + \frac{x}{n}}^{n \text{ times}} \bigg) = E \left(\frac{x}{n} \right)^n,$$

so $E(x/n) = E(x)^{1/n}$. Also, when $m \in \mathbb{N}$, we have $E(-mx)E(mx) = E(-mx + mx) = E(0) = 1$, whence

$$E(-mx) = \frac{1}{E(mx)} = \frac{1}{E(x)^m} = E(x)^{-m}.$$

Now, write $\alpha = m/n$ for some $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. Then,

$$E\left(\frac{m}{n}x\right) = E\left(\frac{x}{n}\right)^m = \left(E(x)^{1/n}\right)^m = E(x)^{m/n}.$$

$\square$

In MATH2050 we defined the number e by

$$e = 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots,$$

we have $E(1) = e$. By Theorem 3.12(b)

$$E(\alpha) = E(\alpha \cdot 1) = E(1)^\alpha = e^\alpha, \quad \alpha \in \mathbb{Q}.$$

Although the left hand side $E(\alpha)$ is well-defined for all real number α, so far we have only defined the right hand side e^α for rational number α. We will shortly use this relation to define e^α for all real α.

From Theorem 3.11(d) and (b), we deduce that E has an inverse function which is continuous from $(0, \infty)$ to $\mathbb{R}$. As E' is always positive, this inverse function is differentiable. We call it the **(natural) logarithmic function** and denote it by $\log x$ or $\ln x$. By translating Theorems 3.11 and 3.12 to the logarithmic function, we have

Theorem 3.13.

(a) $\log : (0, \infty) \to \mathbb{R}$ *is strictly increasing, concave, and satisfies* $\log 1 = 0$, $\lim_{x \to \infty} \log x = \infty$ *and* $\lim_{x \to 0^+} \log x = -\infty$.

(b) $\dfrac{d}{dx}(\log x) = \dfrac{1}{x}, \ \forall \ x > 0.$

(c) $\log(xy) = \log x + \log y, \ \forall \ x, y > 0.$

(d) $\log(x^\alpha) = \alpha \log x, \ \forall \ \alpha \in \mathbb{Q}, \ x > 0.$

Note that (b) implies that the logarithmic function is smooth. Recall that a function f is **concave** is its negative $-f$ is convex.

Proof. (a) follows from Theorems 3.11 and 3.12 and the fact that log is the inverse of the exponential function.
(b) We differentiate the relation $E(\log x) = x$. Applying Chain Rule to the left hand side yields

$$\frac{d}{dx}E(\log x) = E'(\log x)\frac{d}{dx}\log x = E(\log x)\frac{d}{dx}\log x = x\frac{d}{dx}\log x,$$

while the right hand side is equal to 1. Hence (b) holds.

(c) Since E is one-to-one, it suffices to check $E(\log xy) = E(\log x + \log y)$. But then $E(\log xy) = xy$ while $E(\log x + \log y) = E(\log x)E(\log y) = xy$ by Theorem 3.12(a).

(d) Again it suffices to check $E(\log(x^\alpha)) = E(\alpha \log x))$. But this simply means $x^\alpha = E(\alpha \log x)$. Letting $\alpha = n/m$ where $n \in \mathbb{Z}$ and $m \in \mathbb{N}$, by Theorem 3.12(b),

$$
\begin{aligned}
x^\alpha &= (x^n)^{1/m} \\
&= (E(\log x)^n)^{1/m} \\
&= E(n \log x)^{1/m} \\
&= E\left(\frac{n}{m} \log x\right) \\
&= E(\alpha \log x) \ .
\end{aligned}
$$

$\square$

We note that $\log x$ goes to ∞ slower that any positive power, that is, for each positive α, $\lim_{x \to \infty} \log x / x^\alpha = 0$. For, letting $y = \log x$, then $x = E(y)$ and

$$
\lim_{x \to \infty} \frac{\log x}{x^\alpha} = \lim_{y \to \infty} \left(\frac{y^{1/\alpha}}{E(y)}\right)^\alpha = 0.
$$

Now we use the exponential and logarithmic functions to define the real power of a positive number. By Theorem 3.12(b)

$$
E(\alpha) = e^\alpha \ , \quad \alpha \in \mathbb{Q} \ .
$$

Since the RHS only makes sense for rational numbers but the LHS is well-defined for all real number, it is natural to define

$$
e^\alpha = E(\alpha) \ , \quad \alpha \in \mathbb{R} \setminus \mathbb{Q} \ .
$$

Now e^α is well-defined for all real numbers α. Next define the **power (function)** of α, $\alpha \in \mathbb{R}$ to be,

$$
x^\alpha = E(\alpha \log x) \ .
$$

By the chain rule, for each fixed α, x^α is a smooth function on $(0, \infty)$. On the other hand, for each fixed $x > 0$, it is a smooth function in $\alpha \in \mathbb{R}$.

We used to define the rational power by

$$
x^\alpha = (x^n)^{1/m} \ , \quad \alpha = \frac{n}{m}, \ n \in \mathbb{Z}, \ m \in \mathbb{N} \ .
$$

Thus we need to show that our new definition is consistent with the old one for a rational power, that is, $(x^n)^{1/m} = E(n/m \log x)$, but this is already contained in Theorem 3.13(d) (or more precisely in its proof).

We have all the well-known properties of the power function.

Proposition 3.14. *For $x > 0$ and $\alpha \in \mathbb{R}$,*

(a) $\log x^\alpha = \alpha \log x$,

(b) $x^\alpha x^\beta = x^{\alpha+\beta}$,

(c) $(x^\alpha)^\beta = x^{\alpha\beta}$,

(d) $\dfrac{d}{dx} x^\alpha = \alpha x^{\alpha-1}$.

(e) $\dfrac{d}{d\alpha} x^\alpha = \log x \; x^\alpha$.

Proof. We only prove (b) and leave the rest as exercise. Indeed, taking log of both sides of (b), we have

$$
\begin{aligned}
\log(x^\alpha x^\beta) &= \log \big(E(\alpha \log x) \cdot E(\beta \log x) \big) \\
&= \log E(\alpha \log x) + \log E(\beta \log x) \qquad (\text{ by Theorem 3.15(c)}) \\
&= (\alpha + \beta) \log x \\
&= \log E((\alpha + \beta) \log x) \\
&= \log x^{\alpha+\beta} \qquad (\text{by (a)}).
\end{aligned}
$$

$\square$

3.5 Trigonometric Functions

Consider the two problems for the second order differential equation

$$
\frac{d^2 f}{dx^2} + f = 0, \quad f(0) = 1, \; f'(0) = 0. \tag{3.8}
$$

$$
\frac{d^2 f}{dx^2} + f = 0, \quad f(0) = 0, \; f'(0) = 1. \tag{3.9}
$$

Theorem 3.15. *(a) There exists a unique solution $C(x)$ to (3.8) and a unique solution $S(x)$ to (3.9). In fact, they are given respectively by the series*

$$\sum_{j=0}^{\infty} \frac{(-1)^j x^{2j}}{(2j)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots ,$$

and

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1} x^{2j-1}}{(2j-1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots ,$$

where the convergence is uniform on any bounded interval.
(b) $C(x)$ and $S(x)$ are smooth on $\mathbb{R}$ and satisfy $C'(x) = -S(x)$ and $S'(x) = C(x)$.

Proof. We set

$$C(x) = \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j}}{(2j)!},$$

and

$$S(x) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1} x^{2j+1}}{(2j+1)!}.$$

On $[-M, M]$, $M > 0$, we claim that these series converge uniformly. For $C(x)$ we have

$$\left| \frac{(-1)^j x^{2j}}{(2j)!} \right| \le a_j \equiv \frac{M^{2j}}{(2j)!}.$$

It is known that $\sum_j a_j < \infty$. By M-test, the conclusion follows. In particular, $C(x) = \sum_{j=0}^{\infty} (-1)^j x^{2j}/(2j)!$ is convergent for every $x \in \mathbb{R}$. Same results hold for $S(x)$.

Similarly, one can show that the series obtained by termwise differentiating the series defining C and S once converge uniformly on every $[-M, M]$. By Theorem 3.8', it follows that C and S satisfy (3.8) and (3.9) respectively on $[-M, M]$. Clearly, it implies that both functions are smooth. The uniqueness of C and S follows from the lemma below.

$\square$

Lemma 3.16. *Let f and g be two twice differentiable function both satisfying the equation in (3.8) in $\mathbb{R}$. Suppose that $f(x_0) = g(x_0)$ and $f'(x_0) = g'(x_0)$ at some x_0. Then f and g are identical.*

Proof. Although we could imitate what was done in the case of the exponential function, this lemma is most easily proved by the following trick. Letting $h = f - g$, we have

$$\frac{d}{dx} \left(h'^2 + h^2 \right) = 2h'(h'' + h) = 0 ,$$

hence $h'^2 + h^2 = c$ for some constant c whenever h solves the equation in (3.8). When $h(x_0) = h'(x_0) = 0$, the constant c is equal to 0, so h vanishes identically.

$\square$

If g is another solution of (3.8), the $f = g - C$ solves the equation in (3.8) and $f'(0) = f(0) = 0$. Using this lemma we conclude $g \equiv 0$, that is, $g \equiv C$. Similarly one can establish the uniqueness of S.

Proposition 3.17. *We have, for every $x, y \in \mathbb{R}$,*

(a) $C(-x) = C(x), \ S(-x) = -S(x)$;

(b) $C^2(x) + S^2(x) = 1$;

(c) $C(x + y) = C(x)C(y) - S(x)S(y), \ S(x + y) = S(x)C(y) + S(y)C(x)$.

(d) $C'(x) = S(x) , \quad S'(x) = C(x)$.

Proof. (a) This is obvious from the series representation.

(b) This simply follows from $f'^2 + f^2 = 1$ and $f = C, \ f' = S$.

(c) We prove the first identity only. For fixed y, both $C(x + y)$ and $C(x)C(y) - S(x)S(y)$ satisfy the same differential equation $f'' + f = 0$. It follows from uniqueness that they are identical if $C(0 + y) = C(0)C(y) - S(0)S(y)$ and $C'(0 + y) = C'(0)C(y) - S'(0)S(y)$. But this is clearly true.

(d) It follows from the power series representation of both functions. $\square$

Now we come to the periodicity of the trigonometric functions. Recall that a function f on $(-\infty, \infty)$ is called **periodic** if there is some non-zero number T such that $f(x + T) = f(x), \forall x$. The number T is called a **period** of f. As $-T$ is again a period whenever T is a period, so are nT for all $n \in \mathbb{N}$. It is nice to have the concept of the minimal period. A positive number T is called the **minimal period** for f if it is a period and there is not other period in $(0, T)$. In a previous exercise and the midterm exam we showed that for a non-constant continuous periodic function has a minimal period and all other periods are its integer multiples.

Theorem 3.18. *There exists a positive number P such that $C(x + P) = C(x)$ and $S(x + P) = S(x)$ for all $x \in \mathbb{R}$.*

Our proof is very different from the one in [BS].

Proof. We first show that C must vanish somewhere. We know that $C(0) = 1, C'(0) = -S(0) = 0, C''(0) = -1 < 0$ and $C''(x) = -C(x)$. The first three relations tell us that C attains a local maximum at 0 and this maximum is strict. The last relation tells us that C is concave on where $C(x)$ is positive. Therefore, we can find a small positive number x_0 at which $C'(x_0) < 0$ and $C(x_0) > 0$. By concavity, we have

$$\frac{C(x) - C(x_0)}{x - x_0} \leq C'(x_0) < 0,$$

as long as $C(x)$ is non-negative. Assume that C is positive for all $x > 0$. Setting $l(x) = C'(x_0)(x - x_0) + C(x_0)$, this inequality can be expressed as $C(x) \leq l(x), x > 0$, that is, the graph of C lies below the graph of the linear function $l(x)$, which is a straight line. However, $l(x)$ vanishes at $x_1 = -C(x_0)/C'(x_0) + x_0$, contradiction holds.

Let p_0 be the first zero of C in $(0, x_1]$. Since $C'' < 0$ on $[0, p_0)$, $C'(x) < C'(0) = 0$, $C(x)$ strictly decreases from 1 to 0 as x runs from 0 to p_0. We claim that C satisfies the relation $C(2p_0 - x) = -C(x)$. For, letting $\varphi(x) = -C(2p_0 - x)$, we verify that $\varphi(p_0) = 0 = C(p_0), \varphi'(p_0) = C'(p_0)$ and φ satisfies the equation in (3.8). By Lemma 3.16 the claim holds. Using this relation and Proposition 3.17(a),

$$
\begin{aligned}
C(x + 4p_0) &= -C(2p_0 - (x + 4p_0)) \\
&= -C(-x - 2p_0) \\
&= -C(x + 2p_0) \\
&= -C(2p_0 - (-x)) \\
&= C(-x) \\
&= C(x),
\end{aligned}
$$

whence $4p_0$ is a period of C. We take $P = 4p_0$.

$\square$

A remark concerning the proof above, in general, for a given function f on $\mathbb{R}$, the function $\tilde{f}(x) = f(2x_0 - x)$ is the reflection of f with respect to the vertical line $x = x_0$, and $\hat{f}(x) = -f(x)$ is the reflection of f with respect to the x-axis. Thus the function φ is composed of two reflections, first reflecting C with respect to the vertical line $x = p_0$ and then reflecting it with respect to the x-axis.

In fact, the period P given in the above proof is the minimal period of C.

Proposition 3.19. *The period $P = 4p_0$ described in the proof of Theorem 3.20 is the minimal period of C. It follows that C is strictly decreasing on $[0, P/2]$ and strictly increasing on $[P/2, P]$.*

Proof. From what have been proved in the previous theorem especially our claim, we know that C decreases strictly from 1 to -1 and 1 as x runs from 0 to $2p_0$,

and 1 and -1 are respectively the maximum and minimum of C over $[0, 2p_0]$. Using the relation $C(4p_0 - x) = C(-x) = C(x)$, C increases strictly from -1 to 1 as x runs from $2p_0$ to $4p_0$, so 1 and -1 are in fact the maximum and minimum of C over $\mathbb{R}$. In particular, $C(x) \in (-1, 1)$ for $x \in (0, P)$. This shows that $P = 4p_0$ is the minimal period of C. For if there is some $T \in (0, P)$ such that $C(x + T) = C(x)$, then $C(T) = C(0) = 1$, which is impossible.

$\square$

To show that P is in fact equal to $2\pi = 2 \times 3.14159 \cdots$, we recall that 2π has been used to denote the circumference of the unit circle. Using line integrals from advanced calculus, for any plane curve γ, we can find a (regular) parametrization of it, that is, a map $c : [a, b] \rightarrow\rightarrow \mathbb{R}^2$ such that $c(t) = (x(t), y(t))$, $c'(t) \neq (0, 0)$, $c([a, b]) = \gamma$, and $c(t)$ is one-to-one for $t \in [a, b)$. We define the length of c to be the line integral

$$\int_a^b \sqrt{x'^2(t) + y'^2(t)} dt.$$

It can be proved that this line integral is independent of the parametrization c. As a result, we define it to be the length of the plane curve γ. The length of a curve is called the circumference or perimeter of the curve when the curve is closed.

We now *define* 2π to be the perimeter of the unit circle. Analytically it can be evaluated by fixing a regular parametrization of the unit circle. We do it as follows. Consider the special parametrization $\gamma : [0, P] \rightarrow \mathbb{R}^2$ given by $\gamma(t) = (C(t), S(t))$. Clearly, $\gamma' \neq (0, 0)$ so it is regular. Furthermore, it travels along the unit circle exactly once in counterclockwise direction as t increases from 0 to P. To see this let us assume for some $t_1, t_2 \in [0, P), t_1 < t_2, \gamma(t_1) = \gamma(t_2)$, that's, $(C(t_1), S(t_1)) = (C(t_2), S(t_2))$ holds. By applying Lemma 3.16 to the function $C(x + t_1) - C(x + t_2)$ one deduces that $C(x + t_1) = C(x + t_2)$ for all x. Replacing x by $x - t_1$ yields $C(x) = C(x + t_2 - t_1)$, which means $t_2 - t_1$ is also a period of γ. But, $t_2 - t_1 < P$, contradicting the minimality of P. So, γ must be one-to-one on $[0, P)$. Moreover, to show that γ is onto the circle, let (x, y) satisfy $x^2 + y^2 = 1$. As $|x| \leq 1$ and C maps $[0, 2p_0]$ onto $[-1, 1]$, we can find some $t \in [0, 2p_0]$ such that $C(t) = x$. From the relation $C^2 + S^2 = 1$ we see that $S(t) = \pm y$. In case $S(t) = y$, we are done. In case $S(t) = -y$, we take $t_1 = -t + 4p_0 \in [2p_0, P)$ to get $C(t_1) = x$ and $S(t_1) = -S(t) = y$. It follows that

$$2\pi \equiv \int_0^P \sqrt{C'^2(t) + S'^2(t)} \, dt = P.$$

In high school the sine and cosine functions were defined via the angle-side relation in a perpendicular triangle. In some first year class power series expansions

were used to define them. From now on we will abandon these approaches and use the discussion in this section to define the cosine and sine functions. After all, an angle, a side, and a triangle, etc, are geometric other than analytical notions. Abandoned together is the unit "degree". We should use "radian", for instance, $45°$ should be understood as $\pi/4$.

From now on we will take $C(x)$ and $S(x)$ to be our rigorous definition of the cosine and sine functions, and the notations will also be switched back to $\cos x$ and $\sin x$. Other trigonometric functions such as tangent and cotangent can be defined as before in terms of the sine and cosine functions.

One word about the inverse trigonometric functions. Take the cosine function as an example. The cosine function is not monotone, but we know over which intervals it is. In order to have an inverse we need to restrict the function on some interval of length less than or equal to π. In particular, the restriction of the cosine function on $[0, \pi]$ is strictly increasing so it has a continuous inverse which is also differentiable on $(0, \pi)$. We denote this particular inverse by Arccos x. Similarly, the inverse of the sine function over $[-\pi/2, \pi/2]$ is denoted by Arcsin x. In general, an interval is needed to specify when we talk about the inverse trigonometric functions.

In summary, in this section transcendental functions including the exponential, cosine and sine functions have been defined as the solutions to some simple differential equations satisfying simple initial conditions. Specifically these solutions are found in the form of infinite series of functions which converge uniformly on every bounded interval of the real line. Using the uniqueness property in differential equations, this approach yields effortlessly basic properties of these functions. Another nice feature of this approach, which is its feasibility to generalization, may be briefly seen in some exercises. In fact, many so-called "special functions", such as the Bessel functions, Hermite polynomials, Legendre functions,...etc, which are so vital in physics and engineering, are defined as solutions to some second order linear differential equations.